ON THE ASSOCIATED PRIME IDEALS OF LOCAL COHOMOLOGY MODULES DEFINED BY A PAIR OF IDEALS

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ABSTRACT. Let I and J be two ideals of a commutative Noetherian ring R and M be an R-module. For a non-negative integer n it is shown that, if the sets $\mathrm{Ass}_R(\mathrm{Ext}^n_R(R/I, M))$ and $\mathrm{Supp}_R(\mathrm{Ext}^i_R(R/I, H^j_{I,J}(M)))$ are finite for all $i \leq n+1$ and all $j < n$, then so is $\mathrm{Ass}_R(\mathrm{Hom}_R(R/I, H^n_{I,J}(M)))$. We also study the finiteness of $\mathrm{Ass}_R(\mathrm{Ext}^i_R(R/I, H^n_{I,J}(M)))$ for $i = 1, 2$.
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1. Introduction

Let R be a commutative Noetherian ring, I and J be two ideals of R and M be an R-module. For all $i \in \mathbb{N}_0$ the i-th local cohomology functor with respect to (I, J), denoted by $H^i_{I,J}(-)$, defined by Takahashi et. all in [13] as the i-th right derived functor of the (I, J)-torsion functor $\Gamma_{I,J}(-)$, where

$$\Gamma_{I,J}(M) := \{x \in M : I^n x \subseteq Jx \text{ for } n \gg 1\}.$$

This notion coincides with the ordinary local cohomology functor $H^i_I(-)$ when $J = 0$, see [5].

The main motivation for this generalization comes from the study of a dual of ordinary local cohomology modules $H^i_I(M)$ ([11]). Basic facts and more information about local cohomology defined by a pair of ideals can be obtained from [13], [6] and [7].

Hartshorne in [8] proposed the following conjecture:

" Let M be a finitely generated R-module and $\mathfrak{a}$ be an ideal of R. Then $\mathrm{Ext}^i_R(R/\mathfrak{a}, H^j_{\mathfrak{a}}(M))$ is finitely generated for all $i \geq 0$ and $j \geq 0$."

Also, Huneke in [9] raised some crucial problems on local cohomology modules. One of them was about the finiteness of the set of associated prime ideals of the local cohomology modules $H^i_I(M)$.

Although there are some counterexamples to theses conjectures, see [12], but there are some partial positive answers in some special cases too, see for example [3] or [4].

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In this paper, we consider these two problems for local cohomology modules defined by a pair of ideals over not necessary finitely generated modules. In particular, we investigate certain conditions on these modules such that the set of associated prime ideals of $\mathrm{Ext}\,^i_R(R/I, H^j_{I,J}(M))$ is finite.

More precisely, let $n \in \mathbb{N}_0$ and assume that the sets $\mathrm{Ass}_R(\mathrm{Ext}\,^n_R(R/I, M))$ and $\mathrm{Supp}\,_R(\mathrm{Ext}\,^i_R(R/I, H^j_{I,J}(M)))$ are finite for all $i \leq n+1$ and all $j < n$ then, we use a spectral sequence argument to show that $\mathrm{Ass}_R(\mathrm{Hom}\,_R(R/I, H^n_{I,J}(M)))$ is finite, too (Theorem 2.3). Moreover, it is shown that if the sets $\mathrm{Ass}_R(\mathrm{Ext}\,^{n+1}_R(R/I, M))$ and $\mathrm{Supp}\,(\mathrm{Ext}\,^i_R(R/I, H^j_{I,J}(M)))$ are finite for all $i \leq n+2$ and all $j < n$ then, so is $\mathrm{Ass}_R(\mathrm{Ext}\,^1_R(R/I, H^n_{I,J}(M)))$ (Theorem 2.7).

We also present a necessary and sufficient condition for the finiteness of the set $\mathrm{Ass}_R(\mathrm{Ext}\,^2_R(R/I, H^n_{I,J}(M)))$ (Theorem 2.8). These generalize some known results concerning ordinary local cohomology modules.

In [14, 3.6] the authors study the $\underset{M}{\mathrm{grade}}\,\mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Ass}_R(H^t_{I,J}(M))$, where

$$t = inf\{i \in \mathbb{N}_0 : H^i_{I,J}(M) \neq 0\}$$

and M is a finitely generated R-module. But their proof is not correct. Actually, they use the equality $\mathrm{Supp}\,_R(M_x) = \{\mathfrak{p} \in \mathrm{Supp}\,_R(M) : x \notin \mathfrak{p}\}$ which is not true. Here, we also made a correction to this result for not necessary finite modules (Theorem 2.11).

2. Associated prime ideals

In this section, first, we are going to study the set of associated prime ideals of some Ext-modules of local cohomology modules defined by a pair of ideals.

The following relation between associated prime ideals of modules in an exact sequence is frequently used in our results.

Lemma 2.1. *Let $M \to N \to K \to 0$ be an exact sequence of R-modules. Then* $\mathrm{Ass}(K) \subseteq \mathrm{Supp}\,(M) \cup \mathrm{Ass}(N)$.

Proof. Let $\mathfrak{p} \in \mathrm{Ass}(K)$. Assume that $\mathfrak{p} \notin \mathrm{Supp}\,(M)$. Then $M_\mathfrak{p} = 0$ and so $N_\mathfrak{p} \cong K_\mathfrak{p}$. Since $\mathfrak{p}R_\mathfrak{p} \in \mathrm{Ass}_{R_\mathfrak{p}}(N_\mathfrak{p})$, we get $\mathfrak{p} \in \mathrm{Ass}(N)$. □

Next lemma describes a convergence of Grothendieck spectral sequences.

Lemma 2.2. *Let M be an R-module. Then the following convergence of spectral sequences exists*

$$Ext^i_R(R/I, H^j_{I,J}(M)) \overset{i}{\Rightarrow} Ext^{i+j}_R(R/I, M).$$

Proof. It is easy to see that $\mathrm{Hom}\,_R(R/I, \Gamma_{I,J}(M)) = \mathrm{Hom}\,_R(R/I, M)$. Also, for any injective R-module E, $\Gamma_{I,J}(E)$ is an injective R-module, by [13, 3.2] and [5, 2.1.4]. Now, in view of [10, 10.47], the assertion follows. □

The following theorem, which concerns with Hartshorne's problem mentioned in the introduction, is one of the main results in this paper.

Theorem 2.3. *Let n be a non-negative integer and M be an R-module such that* $\mathrm{Ass}_R(\mathrm{Ext}\,^n_R(R/I, M))$ *and* $\mathrm{Supp}\,_R(\mathrm{Ext}\,^i_R(R/I, H^j_{I,J}(M)))$ *are finite for all $i \leq n+1$ and all $j < n$. Then so is $Ass_R(Hom\,_R(R/I, H^n_{I,J}(M)))$.*

Proof. Consider the convergence of spectral sequences in Lemma 2.2 and note that $E_2^{i,j} = 0$ for all $i < 0$. Therefore, for all $2 \leq r \leq n+1$ there exists an exact sequence

$$0 \to E^{0,n}_{r+1} \to E^{0,n}_r \xrightarrow{d^{0,n}_r} E^{r,n+1-r}_r. \tag{2.1}$$

Since, $E_r^{r,n+1-r}$ is a subquotient of $E_2^{r,n+1-r} = \mathrm{Ext}\,^r_R(R/I, H^{n+1-r}_{I,J}(M))$, $\mathrm{Supp}\,_R(E_r^{r,n+1-r})$ is a finite set. So, the above exact sequence implies that $\sharp\mathrm{Ass}_R(E^{0,n}_r) < \infty$ if $\sharp\mathrm{Ass}_R(E^{0,n}_{r+1}) < \infty$. Also, from the fact that $E_2^{i,j} = 0$ for all $j < 0$, we have $E^{0,n}_\infty \cong E^{0,n}_{n+2}$. Therefore, to prove the assertion it is enough to show that $\mathrm{Ass}_R(E^{0,n}_\infty)$ is a finite set.

Using the concept of the convergence of spectral sequences, there exists a bounded filtration

$$0 = \varphi^{n+1}H^n \subseteq \varphi^n H^n \subseteq ... \subseteq \varphi^1 H^n \subseteq \varphi^0 H^n = \mathrm{Ext}\,^n_R(R/I, M)$$

of submodules of $\mathrm{Ext}\,^n_R(R/I, M)$ such that

$$E^{i,n-i}_\infty \cong \varphi^i H^n/\varphi^{i+1}H^n \text{ for all } i = 0, ..., n.$$

Therefore, $E^{n,0}_{n+1} \cong E^{n,0}_\infty \cong \varphi^n H^n$ is a subquotient of $E_2^{n,0} = \mathrm{Ext}\,^n_R(R/I, \Gamma_{I,J}(M))$. So, by assumption, $\mathrm{Supp}\,_R(\varphi^n H^n)$ is a finite set. Now, assume inductively that $\sharp\mathrm{Supp}\,_R(\varphi^i H^n) < \infty$ for all $1 < i \leq n$. Then, since

$$E^{1,n-1}_{n+1} \cong E^{1,n-1}_\infty \cong \varphi^1 H^n/\varphi^2 H^n$$

is a subquotient of $E_2^{1,n-1} = \mathrm{Ext}\,^1_R(R/I, H^{n-1}_{I,J}(M))$, we deduce that $\mathrm{Supp}\,_R(\varphi^1 H^n)$ is finite. But,

$$E^{0,n}_\infty \cong \mathrm{Ext}\,^n_R(R/I, M)/\varphi^1 H^n$$

and Lemma 2.1 implies that $\sharp\mathrm{Ass}_R(E^{0,n}_\infty) < \infty$, as desired.

□

As an immediate consequence of Theorem 2.3, we obtain the following result that is a generalization of [2, 2.3].

Corollary 2.4. *Let M be a finite R-module. Suppose that there is an integer n such that for all $i < n$ the set* $\mathrm{Supp}\,_R(H^i_{I,J}(M))$ *is finite. Then* $\mathrm{Ass}_R(H^n_{I,J}(M))$ *is finite.*

Corollary 2.5. *Let M be a finite R-module and $t = inf\{i | H^i_{I,J}(M) \neq 0\}$ be an integer. Then* $\mathrm{Ass}_R(\mathrm{Hom}\,_R(R/I, H^t_{I,J}(M)))$ *is finite. If in addition,* $\operatorname{grade}_M I = t$*, then for a maximal M-sequence $x_1, ..., x_t$ in I, we have*

$$\mathrm{Ass}_R(\mathrm{Hom}\,_R(R/I, H^t_{I,J}(M))) = \{\mathfrak{p} \in \mathrm{Ass}_R(M/(x_1, .., x_t)M) \cap V(I); \operatorname{grade}_M \mathfrak{p} = t\}.$$

Proof. It is straightforward from Theorem 2.3, [14, 3.10] and [1, 2.6]. □

Corollary 2.6. *Let M be a finite R-module. Suppose that $q = inf\{i : H^i_{I,J}(M)$ is not Artinian $\}$ is an integer, then $\mathrm{Ass}_R(\mathrm{Hom}_R(R/I, H^q_{I,J}(M)))$ is finite.*

In the rest of this paper we consider the set of associated prime ideals of some Ext modules of local cohomology modules defined by a pair of ideals.

Theorem 2.7. *Let n be a non-negative integer and M be an R-module such that $\mathrm{Ass}_R(\mathrm{Ext}^{n+1}_R(R/I, M))$ and $\mathrm{Supp}_R(\mathrm{Ext}^i_R(R/I, H^j_{I,J}(M)))$ are finite for all $i \leq n+2$ and all $j < n$. Then so is $Ass_R(Ext^1_R(R/I, H^n_{I,J}(M)))$.*

Proof. Considering the convergence of the spectral sequences of Lemma 2.2, we have to show that $\mathrm{Ass}_R(E^{1,n}_2)$ is a finite set. Using similar arguments as used in Theorem 2.3, one can see that it is enough to show that $\mathrm{Ass}_R(E^{1,n}_\infty) = \mathrm{Ass}_R(E^{1,n}_{n+2})$ is a finite set.

By the concept of convergence of spectral sequences, there exists a filtration

$$0 = \varphi^{n+2}H^{n+1} \subseteq \varphi^{n+1}H^{n+1} \subseteq ... \subseteq \varphi^1 H^{n+1} \subseteq \varphi^0 H^{n+1} = \mathrm{Ext}^{n+1}_R(R/I, M)$$

of submodules of $\mathrm{Ext}^{n+1}_R(R/I, M)$ such that $E^{i,n+1-i}_\infty \cong \varphi^i H^{n+1}/\varphi^{i+1}H^{n+1}$ for all $i = 0, ..., n+1$. Using the fact that $\sharp\mathrm{Supp}_R(E^{i,j}_2) < \infty$ for all $i \leq n+2$ and all $j < n$ one can see that $\mathrm{Supp}_R(\varphi^i H^{n+1})$ is a finite set for all $i = 2, ..., n+2$. Also, $\sharp\mathrm{Ass}_R(\varphi^1 H^{n+1}) < \infty$. Now, since

$$E^{1,n}_{n+2} \cong E^{1,n}_\infty \cong \varphi^1 H^{n+1}/\varphi^2 H^{n+1},$$

using Lemma 2.1, we have $\sharp\mathrm{Ass}_R(E^{1,n}_\infty) < \infty$, and the result follows. □

The following theorem presents a necessary and sufficient condition for the finiteness of the set $\mathrm{Ass}_R(\mathrm{Ext}^i_R(R/I, H^n_{I,J}(M)))$ when $i = 1, 2$.

Theorem 2.8. *Let n be a non-negative integer and M be an R-module such that the sets $Supp_R(Ext^{n+1}_R(R/I, M))$ and $Supp_R(Ext^i_R(R/I, H^j_{I,J}(M)))$ are finite for all $i \leq n+2$ and all $j < n$. Then $Ass_R(Hom_R(R/I, H^{n+1}_{I,J}(M)))$ is finite if and only if $Ass_R(Ext^2_R(R/I, H^n_{I,J}(M)))$ is finite.*

Proof. ($\Leftarrow$) Again, consider the convergence of spectral sequences of Lemma 2.2 and assume that $Ass_R(E^{2,n}_2)$ is finite. Since $E^{i,j}_2 = 0$ for all $i < 0$ or $j < 0$, using similar arguments as used in Theorem 2.3, one can see that $E^{0,n+1}_\infty \cong E^{0,n+1}_{n+3}$ and in order to prove that $\sharp\mathrm{Ass}_R(E^{0,n+1}_2) < \infty$ we have to show that $\sharp\mathrm{Ass}_R(E^{0,n+1}_\infty) < \infty$.

There exists a filtration

$$0 = \varphi^{n+2}H^{n+1} \subseteq \varphi^{n+1}H^{n+1} \subseteq ... \subseteq \varphi^1 H^{n+1} \subseteq \varphi^0 H^{n+1} = \mathrm{Ext}^{n+1}_R(R/I, M)$$

of submodules of $\mathrm{Ext}^{n+1}_R(R/I, M)$ such that $E^{0,n+1}_\infty \cong \mathrm{Ext}^{n+1}_R(R/I, M)/\varphi^1 H^{n+1}$. Since $\sharp\mathrm{Ass}_R(\mathrm{Ext}^{n+1}_R(R/I, M)) < \infty$ we have $\sharp\mathrm{Ass}_R(E^{0,n+1}_\infty) < \infty$, as desired

($\Rightarrow$) Now, assume that $\mathrm{Ass}_R(\mathrm{Hom}_R(R/I, H^{n+1}_{I,J}(M))) < \infty$ and consider the exact sequence

$$0 \to \mathrm{Ker}\, d_2^{0,n+1} \to E_2^{0,n+1} \xrightarrow{d_2^{0,n+1}} \mathrm{Im}\, d_2^{0,n+1} \to 0.$$

Since $\mathrm{Ker}\, d_2^{0,n+1} = E_3^{0,n+1}$ and $\sharp\mathrm{Supp}_R(E_3^{0,n+1}) < \infty$, in view of Lemma 2.1, we have $\sharp\mathrm{Ass}_R(\mathrm{Im}\, d_2^{0,n+1}) < \infty$. Now, using the exact sequence

$$0 \to \mathrm{Im}\, d_2^{0,n+1} \to E_2^{2,n} \xrightarrow{d_2^{2,n}} E_2^{4,n-1}$$

and the fact that $E_2^{4,n-1} = \mathrm{Ext}^4_R(R/I, H^{n-1}_{I,J}(M))$ has finite support, we have $\sharp\mathrm{Ass}_R(E_2^{2,n}) < \infty$, as desired.

□

Theorem 2.9. *Let n be a non-negative integer and M be an R-module of dimension d, such that $\mathrm{Ass}_R(\mathrm{Ext}^{n+d}_R(R/I, M))$ and $\mathrm{Supp}_R(\mathrm{Ext}^i_R(R/I, H^j_{I,J}(M)))$ are finite for all $i \geq n+1$ and all $j < d$. Then $\mathrm{Ass}_R(\mathrm{Ext}^n_R(R/I, H^d_{I,J}(M)))$ is finite.*

Proof. The method of the proof is similar to the Theorem 2.7, considering [13, 4.7].

□

In the rest of this paper, we study "the grade" of prime ideals $\mathfrak{p} \in \mathrm{Ass}_R(H^t_{I,J}(M))$ on M.

For an R-module M and an ideal $\mathfrak{a}$ of R, the grade of $\mathfrak{a}$ on M is defined by

$$\operatorname*{grade}_M \mathfrak{a} := \inf\{i \in \mathbb{N}_0 : H^i_{\mathfrak{a}}(M) \neq 0\},$$

if this infimum exists, and ∞ otherwise. If M is a finite R-module and $\mathfrak{a}M \neq M$, this definition coincides with the length of a maximal M-sequence in $\mathfrak{a}$ (cf. [5, 6.2.7]).

Also, we shall use the following notations introduced in [13], in which $W(I, J)$ is closed under specialization, but not necessarily a closed subset of $\mathrm{Spec}\,(R)$.

$$W(I, J) := \{\mathfrak{p} \in \mathrm{Spec}\,(R) : I^n \subseteq \mathfrak{p} + J \text{ for some integer } n \geq 1\},$$

and

$$\widetilde{W}(I, J) := \{\mathfrak{a} : \mathfrak{a} \text{ is an ideal of } R \text{and} I^n \subseteq \mathfrak{a} + J \text{ for some integer } n \geq 1\}.$$

The following lemma can be proved using [13, 3.2].

Lemma 2.10. *For any non-negative integer i and R-module M,*

(*i*) $\mathrm{Supp}_R(H^i_{I,J}(M)) \subseteq \bigcup_{\mathfrak{a} \in \widetilde{W}(I,J)} \mathrm{Supp}\,(H^i_{\mathfrak{a}}(M))$.

(*ii*) $\mathrm{Supp}_R(H^i_{I,J}(M)) \subseteq \mathrm{Supp}_R(M) \cap W(I, J)$.

The following theorem was proved in [14, 3.6] under the hypothesis that M is finite. But the proof is not correct. Here we bring an extension and another proof of this theorem.

Theorem 2.11. *Let M be an R-module and $t = inf\{i \in \mathbb{N}_0 : H^i_{I,J}(M) \neq 0\}$ be a non-negative integer. Then for all $\mathfrak{p} \in \operatorname{Ass}_R(H^t_{I,J}(M))$, $\operatorname*{grade}_M \mathfrak{p} = t$.*

Proof. We use induction on t. Let $t = 0$ and $\mathfrak{p} \in \operatorname{Ass}_R(\Gamma_{I,J}(M))$. Then $\mathfrak{p} = (0 :_R x)$ for some $x \in \Gamma_{I,J}(M)$. Hence $x \in \Gamma_{\mathfrak{p}}(M)$ and so $\Gamma_{\mathfrak{p}}(M) \neq 0$.

Now suppose that $t > 0$ and the case $t - 1$ is settled. Let $\mathfrak{p} \in \operatorname{Ass}(H^t_{I,J}(M))$ and consider the exact sequence $0 \to M \to E \to L \to 0$, where $E = E_R(M)$ is the injective envelope of M. Therefore, using [14, 2.2], $H^i_{I,J}(L) \cong H^{i+1}_{I,J}(M)$ for all $i \geq 0$ and we get

$$inf\{i \in \mathbb{N}_0 : H^i_{I,J}(L) \neq 0\} = inf\{i \in \mathbb{N}_0 : H^i_{I,J}(M) \neq 0\} - 1 = t - 1$$

and that $\mathfrak{p} \in \operatorname{Ass}_R(H^{t-1}_{I,J}(L))$. Thus, by inductive hypothesis, $\operatorname*{grade}_L \mathfrak{p} = t-1$. Now, consider the long exact sequence

$$H^{i-1}_{\mathfrak{p}}(M) \to H^{i-1}_{\mathfrak{p}}(E) \to H^{i-1}_{\mathfrak{p}}(L) \to H^i_{\mathfrak{p}}(M).$$

If $t > 1$, then $H^i_{\mathfrak{p}}(M) \cong H^{i-1}_{\mathfrak{p}}(L) = 0$ for all $i < t$ and $H^t_{\mathfrak{p}}(M) \cong H^{t-1}_{\mathfrak{p}}(L) \neq 0$. Thus $\operatorname*{grade}_M \mathfrak{p} = t$.

Let $t = 1$. Then $\Gamma_{\mathfrak{p}}(L) \neq 0$. By the above exact sequence, it is enough to show that $\Gamma_{\mathfrak{p}}(E) = 0$. On the contrary, assume that $\Gamma_{\mathfrak{p}}(E) \neq 0$. Then there exists a non-zero element $x \in E$ and $n \in \mathbb{N}$ such that $\mathfrak{p}^n x = 0$. We may assume that $\mathfrak{p}^n x = 0$ and $\mathfrak{p}^{n-1}x \neq 0$. So, there exists $r \in \mathfrak{p}^{n-1}$ such that $rx \neq 0$. Thus $\mathfrak{p} \subseteq (0 :_R rx)$. On the other hand, by Lemma 2.10,

$$\mathfrak{p} \in \operatorname{Ass}_R(H^1_{I,J}(M)) \subseteq \operatorname{Supp}_R(H^1_{I,J}(M)) \subseteq \bigcup_{\mathfrak{a} \in \widetilde{W}(I,J)} \operatorname{Supp}_R(H^1_{\mathfrak{a}}(M)).$$

So that there exists $\mathfrak{a} \in \widetilde{W}(I, J)$ such that $\mathfrak{a} \subseteq \mathfrak{p}$. Let $m \in \mathbb{N}$ with $I^m \subseteq \mathfrak{a} + J \subseteq \mathfrak{p} + J \subseteq (0 :_R rx) + J$. Hence $rx \in \Gamma_{I,J}(M)$ which contradicts with hypothesis and the choice of rx. Therefore $\Gamma_{\mathfrak{p}}(E) = 0$ and so $\operatorname*{grade}_M \mathfrak{p} = 1$. □

References

[1] M. Aghapournahr, Kh. Ahmadi-Amoli and M. Y. Sadeghi, *The concepts of depth of a pair of ideals (I, J) on modules and (I, J)-Cohen-Macaulay modules*, arXiv:1301.1015v1 [math. AC] 6 Jan 2013.

[2] K. Borna Lorestani, P. Sahandi and T. Sharif, *A note on the Associated Primes of Local Cohomology Modules*, Comm. Alg., **34**(9) (2006) 3409-3412.

[3] M. Brodmann, *Asymptotic behaviour of cohomology: tameness, supports and associated primes*, S. Ghorpade, H. Srinivasan, J. Verma (Eds.), "Commutative Algebra and Algebraic Geometry" Proceedings, Joint International Meeting of the AMS and the IMS on Commutative Algebra and Algebraic Geometry, Bangalore/India, December 17-20, 2003, Contemporary Mathematics Vol **390** (2005) 31-61.

[4] M. P. Brodmann and A. Lashgari Faghani, *A finiteness result for associated primes of local cohomology modules*, Proc. Amer. Math. Soc., **128**(10) (2000) 2851- 2853.

[5] M. P. Brodmann and R. Y. Sharp, *Local cohomology: An algebraic introduction with geometric applications*, Cambridge University Press, 1998.

[6] L. Chu, *Top local cohomology modules with respect to a pair of ideals*, Proc. Amer. Math. Soc., **139** (2011) 777-782.

[7] L. Chu and Q. Wang, *Some results on local cohomology modules defined by a pair of ideals*, J. Math. Kyoto Univ., **49** (2009) , 193-200.
[8] R. Hartshorne, *Affine duality and cofiniteness*, Invent. Math., **9** (1970) 145-164.
[9] C. Huneke, , Free resolutions in commutative algebra and algebraic geometry, Res. Notes Math. 2, Jones and Bartlett, Boston, MA., (1992) 93-108.
[10] J. Rotman, *An introduction to Homological Algebra*, Academic Press, Second Edition, (2009).
[11] P. Schenzel, *Explicit computations around the Lichtenbaum-Hartshorne vanishing theorem*, Manuscripta Math. 78 (1) (1993) 5768.
[12] A. Singh, *P-torsion elements in local cohomology modules.* (English summary) Math. Res. Lett., **7** (2000) 165-176.
[13] R. Takahashi, Y. Yoshino and T. Yoshizawa, *Local cohomology based on a nonclosed support defined by a pair of ideals*, J. Pure Appl. Algebra., **213** (2009) 582-600.
[14] A. Tehranian and A. Pour Eshmanan Talemi, *Cofinitness of local cohomology based on a non-closed spport defiend by a pair of ideals*, Bull. Iranian Math. Soc., **36**(2) (2010) 145-155.

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